Run the Gamut
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Sources

The Diatonic Scale

Thanks to the keyboard family of instruments, our modern musical diatonic scale (do re mi fa so la ti do) is just slightly different from the one in use in the early middle ages; let's call the medieval one "ut re me fa so la ti ut," like Guido d'Arezzo did. (Okay, Guido didn't specify "ti," but we're going to need it.) The medieval one was based strictly on math and physics; the modern one has been adjusted to allow key changes.

Physics of Higher Harmonics

Why is the octave considered a perfect and harmonious interval, but the (medieval) third is not? Why are the fifth and the fourth harmonious, but the sixth isn't? The answer is found in the physics of vibration.

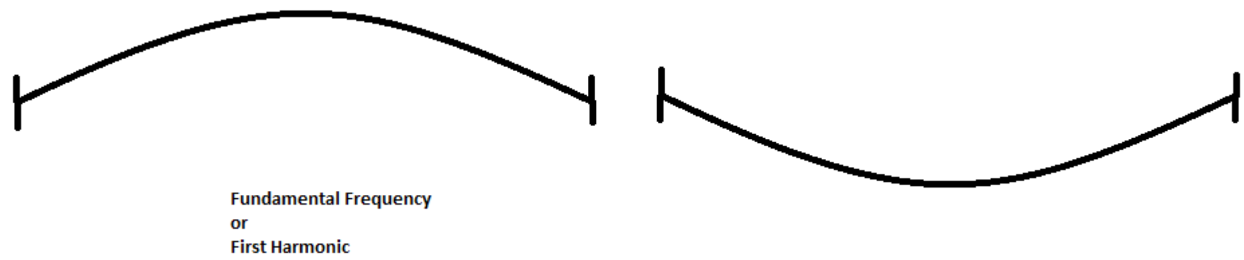
(Intervals are a method of describing the space between two notes.

- Two notes next to each other, like ut-re, re-mi, mi-fa are the interval of a *second*.
- Two notes that have a note between them, like ut-mi, re-fa, and mi-so are the interval of a *third*.
- Three notes between = *fourth*, four notes between = *fifth*, and so on.
- Two notes with seven notes between – from ut to ut, or re to re – are called the *octave* rather than an eight.

The medieval theorists also had Greek names for some of these intervals, like *diapason* for octave, but let's not get hung up on terminology.)

If you pluck a taut string and look at it carefully, you should see that it is vibrating to make a point-ended ellipse shape. It is actually transitioning between a shape with a curved arc up, and one with the curved arc down:

First this Then this



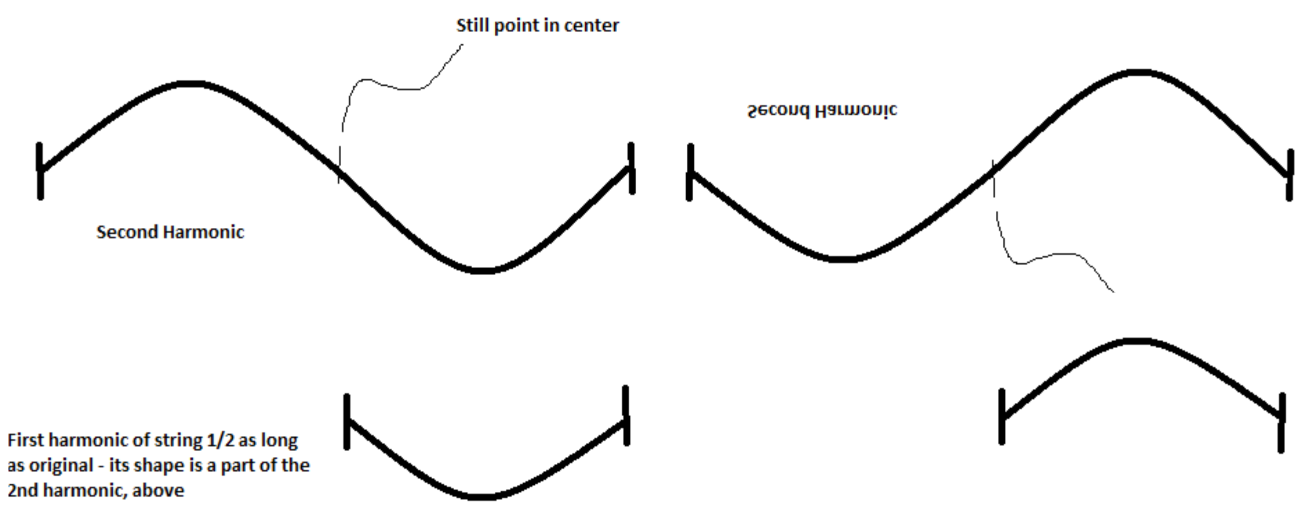
That is a **standing wave**. The speed at which the string is vibrating – the number of shakes back and forth per second that it takes to create that wave – is its **fundamental frequency**. (Also called the *first harmonic*.)

The **fundamental frequency is the note** that you hear. Most of the energy of your pluck goes into this vibration. If this were a tuning fork instead of a vibrating string, this would be the only sound created – a pure tone at this frequency.

But it is a vibrating string, and your pluck will excite other higher harmonics. These are smaller standing waves in the string.

Fun fact about standing waves in vibrating strings: The product of (wavelength x frequency) in a given string must remain constant. So when we get two standing waves, each **half** the wavelength of the wave shown above, the frequency has to **double**.

Here is a picture of the second harmonic of our string. At this frequency – being shaken twice as fast as before – two standing waves are created. This wave pattern gets the next most energy after the fundamental.



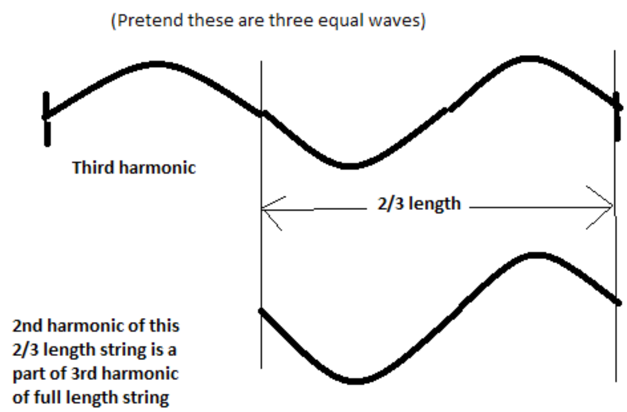
So if I had a second, shorter string – a string half the length of the original string – **its fundamental frequency looks like part of the second harmonic of the original string.**

When the original, longer string is plucked, its second harmonic is happening. You don't necessarily hear it – it's drowned out by the fundamental. But it's there. And this second, shorter string's fundamental frequency – the speed at which it is vibrating – is the **same** as that second harmonic.

So when these two strings are plucked at the same time, they sound extremely harmonious. One is included in the other.

The second, shorter string – the string half the length of the original string – is **sounding one octave up** from the original string. It was considered the most perfectly harmonious interval.

If we vibrate the original string even fast, three times as fast as the fundamental frequency, we will see three standing waves:



This is the third harmonic. It gets less energy than the fundamental and the second harmonic.

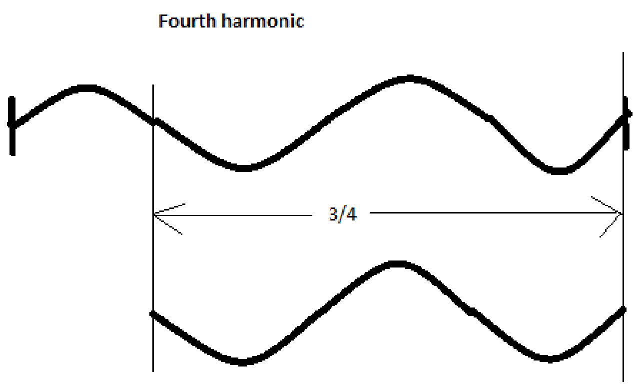
As before, we can find a new, shorter string that's hiding inside of the original string. A string 2/3 of the length of the original string is in there, vibrating at its second harmonic. (There are two standing waves in the shorter string, so that is the second harmonic.)

The frequency – number of shakes per second – of the third harmonic of the original string and the second harmonic of the shorter string is the same.

If we slow the frequency down by half, the shorter string will vibrate at its fundamental frequency. Like our original string, even when the shorter string is sounding its fundamental frequency, it is also sounding its second harmonic. It fits right inside the third harmonic of the original string, and so they sound harmonious together.

This string that's 2/3 the length of the original string sounds a note a **fifth** up from the original string. It was considered the second most harmonious interval.

We can do this again, vibrating the original string four times as fast as its fundamental frequency, making four standing waves:



Third harmonic of this string 3/4 the length of the original is contained in the orginal's 4th harmonic

And again, we can find another shorter string, this one ¾ the length of the original. Its third harmonic (three standing waves) happens at the same frequency as the original's fourth harmonic. And again, because in this way it is "hiding inside" the original string, when they sound together, the interval is harmonious.

If we slow the frequency so that the 3/ 4 length string is vibrating at its fundamental frequency, it will be a **fourth** up from the original string. And the fourth was – guess what? - the third most harmonious interval.

The early medieval theorists recognized the octave, the fifth, the fourth, and the double octave, the octave + a fifth, and the octave + a fourth as harmonious intervals. This is why, even if they didn't know it.

So that gets us ut, fa, so and ut. Where's the rest of the scale?

Pythagoras and the Four Hammers

A version of this (apocryphal) story is found in John Cotton's "On Music," c. 1150, although John doesn't get into the specific weights of the hammers.

One day, Pythagoras (yes, the $a^2 + b^2 = c^2$ guy) was walking through the marketplace, thinking hard about musical scales. As we saw above, they are related to geometry – halving a string raises its tone by an octave, for example. His thoughts were interrupted by the sound of two blacksmiths at work. But it was a serendipitous interruption, as he realized that sometimes, the sound of the two hammers striking the anvil was harmonious, and sometimes it was not. He went over to investigate.

It turned out that, between the two of them, the smiths had four hammers of different sizes: 6 pounds, 8 pounds, 9 pounds, and 12 pounds. As they alternated which ones they were using, Pythagoras found:

- The 6-lb and 12-lb hammers together sounded wonderful. One was half the weight of the other – just like a string half the length of another. This was the octave.
- The 8-lb and 12-lb hammers together also sounded very good. One was 2/3 the weight of the other – this was the fifth, just as we saw above.
- So you should not be surprised to learn that the 9-lb hammer, which was 3/ 4 the weight of the 12-lb hammer, made a harmonious fourth together.
- But the 8-lb and the 9-lb hammer together were terrible!
- (The story doesn't say about the 6-lb and 8-lb, but it would sound good at a fourth: 6/8 = 3/ 4. And the 6-lb and the 9-lb would have sounded good at a fifth: 6/9 = 2/3)

So Pythagoras was in our same position: he had four notes, but not an entire scale. What to do?

Well, between the two of them, the 8-lb and 9-lb hammers defined a small "distance" between two notes. This distance is called a **tone** (also whole step). That distance was a part of this perfect sequence of notes. Could he use it to fill in the rest of the scale?

"So," the 8-lb hammer, is 8/9 of "fa," the 9-lb hammer. Or: if "fa" were a string 9 inches long, we would divide it into nine parts (1 inch each); eight of those parts would make the "so" string, 8 inches long. If the "fa" string were 18 inches long, we'd divide it into nine parts (2 inches each) and "so" would be eight of those – 16 inches. It's a ratio, not a simple fixed length we can add or subtract.

Measure the length of the "ut" string; divide by 9 and multiply the result by 8. This gets you the "re" string.

Measure the length of the "re" string; divide by 9 and multiply the result by 8. This gets you the "mi" string.

Measure the length of the "mi" string; divide by 9 and multiply the result by 8. **This does not get you the "fa" string**. It gets you somewhere between "fa" and "so." Pythagoras sort of shrugged and tossed this "note one step up from mi" out. He just allowed the distance between "mi" and "fa" to be what it was – a **semitone**, or a space less than a tone.

"Fa" and "so" are already defined.

Measure the length of the "so" string; divide by 9 and multiply the result by 8. This gets "la."

Measure "la," divide by 9 and multiply the result by 8. This gets "ti."

"Ti," like "mi," is closer than a tone to the next note, the high "ut." If we measure/divide/multiply, we'd end up between "ut" and "re." So the distance between "ti" and high "ut" is another semitone.

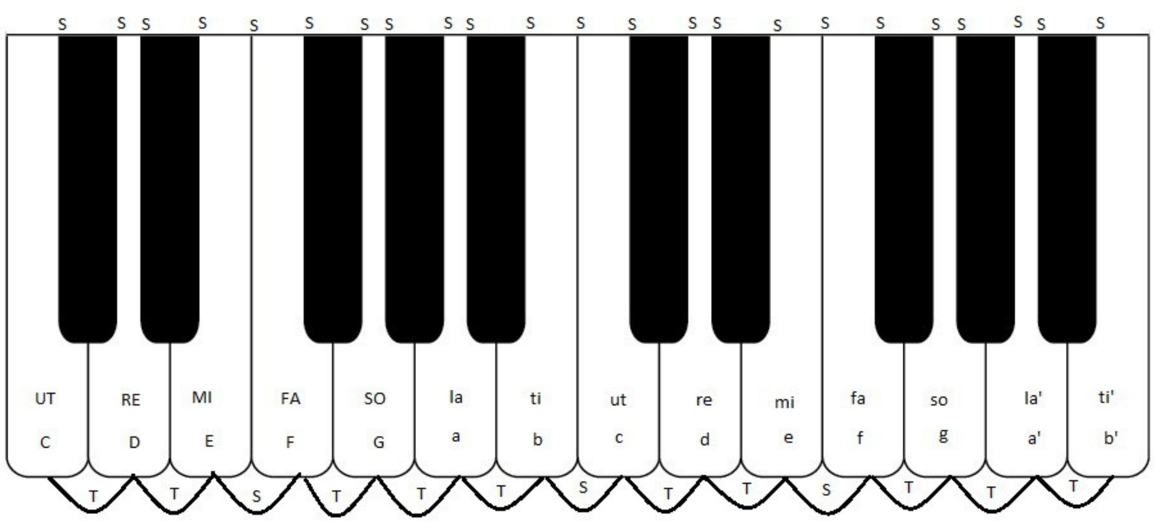
Counting it all up, the pattern of tones and semitones on the diatonic scale is Tone – Tone – Semitone – Tone – Tone – Tone – Semitone.

So we've seen are ratios of re, mi, la and ti to the string that comes just before it. Let's look at the ratios of each string to low "ut."

Ut: Ut	1	
Ut: Re	8/9	
Ut: Mi	64/81	(8/9 of 8/9)
Ut: Fa	3/ 4	
Ut: So	2/3	
Ut: La	16/27	(8/9 of 2/3)
Ut: Ti	128/243	(8/9 of 16/27)
Ut: Hi Ut	1/ 2	

Consider: 1/ 2, 2/3, and 3/ 4 were harmonious intervals. The whole tone, at 8/9, was dissonant.

Is it any wonder that the third, at 64/81, was also dissonant? That is not a geometric ratio which inspires confidence. And in fact, with only a few examples like the St. Magnus Hymn, we don't see parallel thirds featuring prominently in medieval harmonies. (They may occur in passing.)



DIY Diatonic Scale

All of these geometric divisions can be done with a straight edge and compass. For the sake of time, we'll just use a ruler and division. Also, this is not the medieval process, which generates the entire gamut (all the notes from a low G (gamma) to a c" (ut)). The same scale results; I'm just having you do the measurements in a different order so it matches the same order of creating the notes that we've done so far.

On the paper provided, use the ruler provided to draw a 30cm long line. Place one end of the line up against the edge of the paper. Mark the other end "ut".

Divide the line in half. Mark the halfway point as **the octave, "ut' "**

** Making this a little simpler **

To get “so,” the string 2/3 the length of our starting string, we could measure (30cm), divide by 3 (10cm) and then multiply by 2 (20cm). We would then measure from the end of the line at the edge of the paper 20cm and mark “re.”

OR we can measure, divide by 3, and measure up from where we marked “ut.” A mark 1/3 from the “ut” end of the line is at the same place as 2/3 from the other end.

This doesn't make a big deal now, but it'll be much easier to divide by 9 and go that far up from a mark than to divide by 9 and then multiply by 8.

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So divide the length of the line by 3, measure up that far from “ut,” and mark **the fifth, “so.”**

Divide the length by 4; measure up that far from “ut” and mark the fourth, **“fa.”** That is the last perfect interval we get.

We already measured out the length of “ut” (30cm). Divide it by 9, and measure that far up from “ut.” Mark that as **“re,”** one whole tone up.

Measure from “re” to the end of the line. Divide this by 9, and measure that far up **from where you marked “re.”** This is **“mi.”**

Don't measure from “mi” to the end, unless you want to prove to yourself that a whole step up from “mi” is in a no man's land between “fa” and “so.”

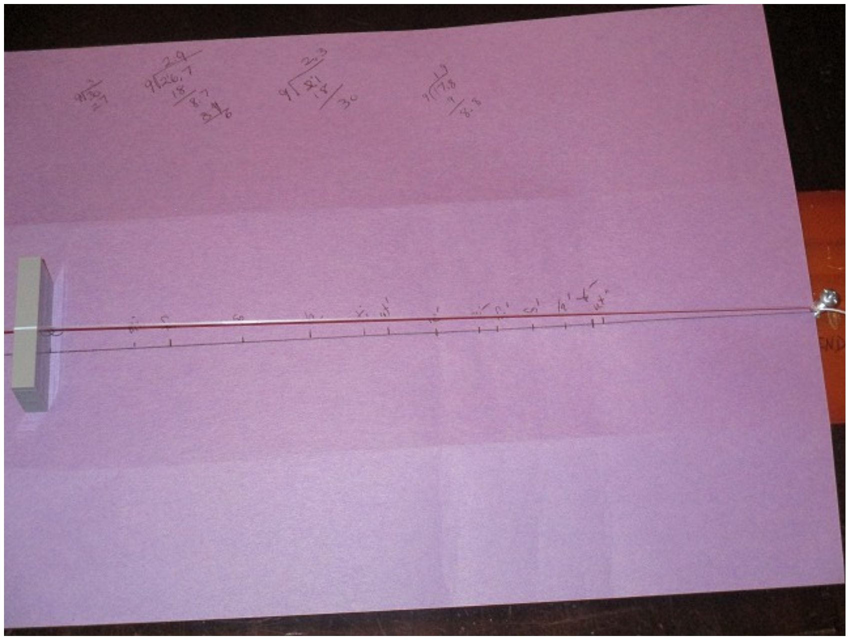
We already derived “fa” and “so.”

Measure from “so” to the end. Divide by 9; measure this far up from “so” and mark “la.”

Measure from “la” to the end. Divide by 9; measure this far up from “la” and mark “ti.”

We already have “ut' ”. If you want, you can get the notes an octave up from each of the ones you've found by measuring each length, dividing it in half, and marking the new note.

It should end up looking something like this:



Playing on the Monochord

Now we'll hear the scales we derived. Place the end of your line (the one at the edge of the paper) against the screw marked END on the monochord.   Place the front edge of the moveable bridge on the mark you made for “ut.” Pluck the string to hear the note. Now slide the bridge forward until the front edge is on the mark for “re.” Pluck again. Continue on to hear all your notes.

That was a lot of time spent on “do re mi.”

Yeah, it was. But the diatonic scale is really the backbone of medieval music theory. The fact that we think we know it so well, but that it's actually a little different today than it was back then, makes it a little tricky. I think it's worth the time to really understand what's going on with where the notes we use are coming from.

References

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